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# Some new local fractional integral inequalities

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**Abstract**

In this study, several new inequalities of local fractional integrals are presented.

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## 1 Introduction

In 1882, Chebyshev [3] proved the following inequality:

**Theorem 1.1.** Let $f$ and $g$ be two integrable functions in $[0,1]$ .If both functions are simultaneously increasing or decreasing for the same values of $x$ in $[0,1]$, then

$$\int_0^1 f(x)g(x)dx \geq \int_0^1 f(x)dx \int_0^1 g(x)dx \tag{1.1}$$

If one function is increasing and the other is decreasing for the same values of $x$ in $[0,1]$, then (1.1) reverses.

Also, author gave the following inequality:

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty , \tag{1.2}$$

where $f,g : [a,b] \to \mathbb{R}$ are absolutely continuous function, whose first derivatives $f'$ and $g'$ are bounded,

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{1.3}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = ess\sup_{t\in[a,b]} |p(t)|$ . In the last years, many papers were deveted to the generalization of the inequalities (1.1) and (1.2), we can mention the works [1], [2], [5]-[8], [11]-[13], [19].

The purpose of this paper is to obtain some local fractional integral inequalities similar to inequalitt (1.1). This paper is divided into the following three sections. In Section 2, we give the definitions of the local fractional derivatives and local fractional integral and introduce several useful notations on fractal sapace used our main results. In Section 3, the main result is presented.

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## 2 Preliminaries

Recall the set $R^{\alpha}$ of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [20, 21] and so on.

Recently, the theory of Yang's fractional sets [20] was introduced as follows.

For $0<\alpha\leq 1$, we have the following $\alpha$-type set of element sets:

$Z^{\alpha}$ : The $\alpha$-type set of integer is defined as the set $\{0^{\alpha},\pm 1^{\alpha},\pm 2^{\alpha},...,\pm n^{\alpha},...\}$.

$Q^{\alpha}$ : The $\alpha$-type set of the rational numbers is defined as the set $\{m^{\alpha}=\left(\frac{p}{q}\right)^{\alpha}:p,q\in Z,\, q\neq 0\}$.

$J^{\alpha}$ : The $\alpha$-type set of the irrational numbers is defined as the set $\{m^{\alpha}\neq\left(\frac{p}{q}\right)^{\alpha}:p,q\in Z,$ $q\neq 0\}$.

$R^{\alpha}$ : The $\alpha$-type set of the real line numbers is defined as the set $R^{\alpha}=Q^{\alpha}\cup J^{\alpha}$.

If $a^{\alpha},b^{\alpha}$ and $c^{\alpha}$ belongs the set $R^{\alpha}$ of real line numbers, then

(1) $a^{\alpha}+b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set $R^{\alpha}$;

(2) $a^{\alpha}+b^{\alpha}=b^{\alpha}+a^{\alpha}=(a+b)^{\alpha}=(b+a)^{\alpha}$;

(3) $a^{\alpha}+(b^{\alpha}+c^{\alpha})=(a+b)^{\alpha}+c^{\alpha}$;

(4) $a^{\alpha}b^{\alpha}=b^{\alpha}a^{\alpha}=(ab)^{\alpha}=(ba)^{\alpha}$;

(5) $a^{\alpha}\left(b^{\alpha}c^{\alpha}\right)=\left(a^{\alpha}b^{\alpha}\right)c^{\alpha}$;

(6) $a^{\alpha}\left(b^{\alpha}+c^{\alpha}\right)=a^{\alpha}b^{\alpha}+a^{\alpha}c^{\alpha}$;

(7) $a^{\alpha}+0^{\alpha}=0^{\alpha}+a^{\alpha}=a^{\alpha}$ and $a^{\alpha}1^{\alpha}=1^{\alpha}a^{\alpha}=a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

**Definition 2.1.** [20] A non-differentiable function $f:R\rightarrow R^{\alpha}$, $x\rightarrow f(x)$ is called to be local fractional continuous at $x_0$, if for any $\varepsilon>0$, there exists $\delta>0$, such that

$$|f(x)-f(x_0)|<\varepsilon^{\alpha}$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval $(a,b)$, we denote $f(x)\in C_{\alpha}(a,b)$.

**Definition 2.2.** [20] The local fractional derivative of $f(x)$ of order $\alpha$ at $x=x_0$ is defined by

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\rightarrow x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{(x-x_0)^{\alpha}},$$

where $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(\alpha+1)\left(f(x)-f(x_0)\right)$.

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^{\alpha}...D_x^{\alpha}}^{k+1\text{ times}}f(x)$ for any $x\in I\subseteq R$, then we denoted $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,...$

**Definition 2.3.** [20] Let $f(x)\in C_{\alpha}\left[a,b\right]$. Then the local fractional integral is defined by,

$${}_aI_b^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\rightarrow 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha},$$

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with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, ..., \Delta t_{N-1}\}$, where $[t_j, t_{j+1}]$, $j = 0, ..., N-1$ and $a = t_0 < t_1 < ... < t_{N-1} < t_N = b$ is partition of interval $[a, b]$.

Here, it follows that ${}_aI_b^\alpha f(x) = 0$ if $a = b$ and ${}_aI_b^\alpha f(x) = -{}_bI_a^\alpha f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_aI_x^\alpha f(x)$, then we denoted by $f(x) \in I_x^\alpha [a, b]$.

**Lemma 2.4.** [20]

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha [a, b]$, then we have

$${}_aI_b^\alpha f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha [a, b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha [a, b]$, then we have

$${}_aI_b^\alpha f(x) g^{(\alpha)}(x) = f(x) g(x)|_a^b - {}_aI_b^\alpha f^{(\alpha)}(x) g(x).$$

**Lemma 2.5.** [20] We have

i) $\dfrac{d^\alpha x^{k\alpha}}{dx^\alpha} = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}$;

ii) $\dfrac{1}{\Gamma(\alpha+1)} \int\limits_a^b x^{k\alpha} (dx)^\alpha = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right), \ k \in R.$

For more information and recent developments on local fractional theory, please refer to [4], [9], [10], [14]-[18], [20]-[27].

# 3 Main Results

**Theorem 3.1.** Let $f, g : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ be two synchronous mapping such that $f, g \in I_x^\alpha [a, b]$ and let $h : [a, b] \to \mathbb{R}^\alpha$ be non-negative such that $h \in I_x^\alpha [a, b]$. Then, we have the following inequality for local fractional integrals

$$\frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \, {}_aI_b^\alpha (fgh)(x) + \left[{}_aI_b^\alpha (fg)(x)\right]\left[{}_aI_b^\alpha h(x)\right] \tag{3.1}$$

$$\geq \left[{}_aI_b^\alpha (fh)(x)\right]\left[{}_aI_b^\alpha g(x)\right] + \left[{}_aI_b^\alpha (f)(x)\right]\left[{}_aI_b^\alpha (gh)(x)\right].$$

*Proof.* Since $f$ and $g$ are synchronous functions on $[a, b]$, for any $x, y \in [a, b]$, we have

$$[f(x) - f(y)][g(x) - g(y)][h(x) + h(y)] \geq 0 \tag{3.2}$$

and from (3.2) we get,

$$\frac{1}{\Gamma^2(1+\alpha)} \int\limits_a^b \int\limits_a^b [f(x) - f(y)][g(x) - g(y)][h(x) + h(y)] (dy)^\alpha (dx)^\alpha \geq 0. \tag{3.3}$$

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On the other hand, by using the local fractional integrals, we have

$$\frac{1}{\Gamma^2(1+\alpha)}\int_a^b\int_a^b [f(x)-f(y)][g(x)-g(y)][h(x)+h(y)](dy)^\alpha(dx)^\alpha \tag{3.4}$$

$$= \frac{1}{\Gamma^2(1+\alpha)}\int_a^b\int_a^b [f(x)g(x)h(x)-f(x)g(y)h(x)-f(y)g(x)h(x)$$

$$+f(y)g(y)h(x)+f(x)g(x)h(y)$$

$$-f(x)g(y)h(y)-f(y)g(x)h(y)+f(y)g(y)h(y)](dy)^\alpha(dx)^\alpha$$

$$= \frac{2^\alpha(b-a)^\alpha}{\Gamma^2(1+\alpha)}\int_a^b f(x)g(x)h(x)(dx)^\alpha$$

$$-2^\alpha\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)h(x)(dx)^\alpha\right)\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b g(x)(dx)^\alpha\right)$$

$$-2^\alpha\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^\alpha\right)\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b g(x)h(x)(dx)^\alpha\right)$$

$$+2^\alpha\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(dx)^\alpha\right)\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b h(x)(dx)^\alpha\right)$$

$$= \frac{2^\alpha(b-a)^\alpha}{\Gamma(1+\alpha)}\ {}_aI_b^\alpha(fgh)(x)+2^\alpha\left[{}_aI_b^\alpha(fg)(x)\right]\left[{}_aI_b^\alpha h(x)\right]$$

$$-2^\alpha\left[{}_aI_b^\alpha(fh)(x)\right]\left[{}_aI_b^\alpha g(x)\right]-2^\alpha\left[{}_aI_b^\alpha f(x)\right]\left[{}_aI_b^\alpha(gh)(x)\right].$$

That is, from (3.3) and (3.4), we obtain

$$\frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\ {}_aI_b^\alpha(fgh)(x)+\left[{}_aI_b^\alpha(fg)(x)\right]\left[{}_aI_b^\alpha h(x)\right]$$

$$-\left[{}_aI_b^\alpha(fh)(x)\right]\left[{}_aI_b^\alpha(h)(x)\right]-\left[{}_aI_b^\alpha(f)(x)\right]\left[{}_aI_b^\alpha(gh)(x)\right]\geq 0$$

which completes the proof. Q.E.D.

**Corollary 3.2.** Under assumption of Theorem 3.1 with $h(x)\equiv 1^\alpha$, then we have

$$\frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\ {}_aI_b^\alpha(fg)(x)\geq\left[{}_aI_b^\alpha f(x)\right]\left[{}_aI_b^\alpha g(x)\right]. \tag{3.5}$$

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**Theorem 3.3.** Let $f, g : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}^{\alpha}$ be a mapping such that $f, g \in C_{\alpha}[a, b]$. Then, we have the following inequality for local fractional integrals

$$\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\left[ {}_{a}I_{b}^{\alpha}\left(f^{2}\right)(x) + {}_{a}I_{b}^{\alpha}\left(g^{2}\right)(x)\right] \geq 2^{\alpha}\left[ {}_{a}I_{b}^{\alpha}f(x)\right]\left[ {}_{a}I_{b}^{\alpha}g(x)\right]. \tag{3.6}$$

*Proof.* Since

$$[f(x) - g(y)]^{2} \geq 0$$

then, we have

$$\frac{1}{\Gamma^{2}(1+\alpha)}\int_{a}^{b}\int_{a}^{b}[f(x) - g(y)]^{2}(dy)^{\alpha}(dx)^{\alpha} \geq 0. \tag{3.7}$$

Also, we get

$$\frac{1}{\Gamma^{2}(1+\alpha)}\int_{a}^{b}\int_{a}^{b}[f(x) - g(y)]^{2}(dy)^{\alpha}(dx)^{\alpha} \tag{3.8}$$

$$= \frac{1}{\Gamma^{2}(1+\alpha)}\int_{a}^{b}\int_{a}^{b}\left[f^{2}(x) - 2^{\alpha}f(x)g(y) + g^{2}(y)\right](dy)^{\alpha}(dx)^{\alpha}$$

$$= \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\left[ {}_{a}I_{b}^{\alpha}\left(f^{2}\right)(x) + {}_{a}I_{b}^{\alpha}\left(g^{2}\right)(x)\right] - \left[ {}_{a}I_{b}^{\alpha}f(x)\right]\left[ {}_{a}I_{b}^{\alpha}g(x)\right].$$

If we combine (3.7) and (3.8), then we obtain the required result. Q.E.D.

**Corollary 3.4.** Under assumption of Theorem 3.3 with $f(x) \equiv g(x)$ for all $x \in [a, b]$, then we have

$$\frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\, {}_{a}I_{b}^{\alpha}f^{2}(x) \geq \left[ {}_{a}I_{b}^{\alpha}f(x)\right]^{2}. \tag{3.9}$$

**Theorem 3.5.** Under assumption of Theorem 3.3, we have the following inequality

$$\left[ {}_{a}I_{b}^{\alpha}f^{2}(x)\right]\left[ {}_{a}I_{b}^{\alpha}g^{2}(x)\right] \geq \left[ {}_{a}I_{b}^{\alpha}fg(x)\right]^{2}. \tag{3.10}$$

*Proof.* Since

$$[f(x)g(y) - f(y)g(x)]^{2} \geq 0$$

then, we have

$$\frac{1}{\Gamma^{2}(1+\alpha)}\int_{a}^{b}\int_{a}^{b}[f(x)g(y) - f(y)g(x)]^{2}(dy)^{\alpha}(dx)^{\alpha} \geq 0. \tag{3.11}$$

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Moreover, we have

$$\begin{aligned}
&\frac{1}{\Gamma^2(1+\alpha)}\int_a^b\int_a^b [f(x)g(y)-f(y)g(x)]^2 (dy)^\alpha (dx)^\alpha \\
&= \frac{1}{\Gamma^2(1+\alpha)}\int_a^b\int_a^b \left[f^2(x)g^2(y)-2^\alpha f(x)g(y)f(y)g(x)+f^2(y)g^2(x)\right] (dy)^\alpha (dx)^\alpha \\
&= 2^\alpha \left[ {}_aI_b^\alpha f^2(x)\right]\left[ {}_aI_b^\alpha g^2(x)\right] - 2^\alpha \left[ {}_aI_b^\alpha fg(x)\right]^2 .
\end{aligned}$$

This completes the proof. Q.E.D.

**Remark 3.6.** If we choose $g(x)\equiv 1^\alpha$ for all $x\in[a,b]$, then the inequality (3.10) reduces (3.9).

**Theorem 3.7.** Let $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}^\alpha$ be a mapping such that $f\in D_\alpha(a,b)$ and $f^{(\alpha)}\in C_\alpha[a,b]$. Then, we have the following inequality

$$\left|f(b)-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(x)\right|\le\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left\|f^{(\alpha)}\right\|_\infty (b-a)^\alpha$$

where $\left\|f^{(\alpha)}\right\|_\infty$ is defined by

$$\left\|f^{(\alpha)}\right\|_\infty=\sup_{t\in[a,b]}\left|f^{(\alpha)}(t)\right|.$$

*Proof.* From the hypothesis, we have the following identity

$$f(b)-f(x)=\frac{1}{\Gamma(1+\alpha)}\int_x^b f^{(\alpha)}(t)\,(dt)^\alpha . \tag{3.12}$$

Taking the modulus in (3.12), for all $x\in[a,b]$, we have

$$\begin{aligned}
|f(b)-f(x)| &\le \frac{1}{\Gamma(1+\alpha)}\int_x^b \left|f^{(\alpha)}(t)\right| (dt)^\alpha \\
&\le \frac{\left\|f^{(\alpha)}\right\|_\infty}{\Gamma(1+\alpha)}(b-x)^\alpha .
\end{aligned}$$

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Therefore,

$$\left|\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}f(x)(dx)^{\alpha}-\frac{f(b)}{\Gamma(1+\alpha)}\right|$$

$$\begin{aligned}
&= \frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}[f(x)-f(b)](dx)^{\alpha}\\
&\leq \frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}|f(b)-f(x)|(dx)^{\alpha}\\
&\leq \frac{\|f^{(\alpha)}\|_{\infty}}{(b-a)^{\alpha}\Gamma^{2}(1+\alpha)}\int_{a}^{b}(b-x)^{\alpha}(dx)^{\alpha}\\
&= \frac{\|f^{(\alpha)}\|_{\infty}}{\Gamma(1+2\alpha)}(b-a)^{2\alpha}.
\end{aligned}$$

This completes the proof. Q.E.D.

**Theorem 3.8.** Let $f,g:[a,b]\subseteq\mathbb{R}\to\mathbb{R}^{\alpha}$ be two functions such that $f,g\in D_{\alpha}(a,b)$ and $f^{(\alpha)},g^{(\alpha)}\in C_{\alpha}[a,b]$. Then, we have the following inequality

$$\left|f(b)\ {}_{a}I_{b}^{\alpha}g(x)+g(b)\ {}_{a}I_{b}^{\alpha}f(x)-2^{\alpha}\ {}_{a}I_{b}^{\alpha}(fg)(x)\right|$$

$$\leq \frac{1}{\Gamma^{2}(1+\alpha)}\int_{a}^{b}\left[\left\|f^{(\alpha)}\right\|_{\infty}g(x)+\left\|g^{(\alpha)}\right\|_{\infty}f(x)\right](b-x)^{\alpha}(dx)^{\alpha}.$$

*Proof.* From the hypothesis, we have the following identities

$$f(b)-f(x)=\frac{1}{\Gamma(1+\alpha)}\int_{x}^{b}f^{(\alpha)}(t)(dt)^{\alpha} \tag{3.13}$$

and

$$g(b)-g(x)=\frac{1}{\Gamma(1+\alpha)}\int_{x}^{b}g^{(\alpha)}(t)(dt)^{\alpha}. \tag{3.14}$$

Multiplying both sides of (3.13) and (3.14) by $g(x)$ and $f(x)$ respectively and adding the resulting

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identities, we have

$$f(b)g(x)+g(b)f(x)-2^{\alpha}f(x)g(x) \tag{3.15}$$

$$= \frac{g(x)}{\Gamma(1+\alpha)}\int_x^b f^{(\alpha)}(t)\,(dt)^{\alpha}+\frac{f(x)}{\Gamma(1+\alpha)}\int_x^b g^{(\alpha)}(t)\,(dt)^{\alpha}.$$

Integrating the both sides of equality (3.15) with respect to $x$ over $[a,b]$ and using the properties of modulus, we get

$$\left|f(b)\ {}_aI_b^{\alpha}g(x)+g(b)\ {}_aI_b^{\alpha}f(x)-2^{\alpha}\ {}_aI_b^{\alpha}(fg)(x)\right|$$

$$\leq \frac{1}{\Gamma^2(1+\alpha)}\int_a^b\left[|g(x)|\int_x^b\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}+|f(x)|\int_x^b\left|g^{(\alpha)}(t)\right|(dt)^{\alpha}\right](dx)^{\alpha}$$

$$\leq \frac{1}{\Gamma^2(1+\alpha)}\int_a^b\left[\left\|f^{(\alpha)}\right\|_{\infty}|g(x)|\,(b-x)^{\alpha}+\left\|g^{(\alpha)}\right\|_{\infty}|f(x)|\,(b-x)^{\alpha}\right](dx)^{\alpha}$$

which completes the proof. Q.E.D.

**Corollary 3.9.** Under assumption of Theorem 3.8 with $g(x)\equiv 1^{\alpha}$, we have

$$\left|f(b)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}(f)(x)\right|\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left\|f^{(\alpha)}\right\|_{\infty}(b-a)^{\alpha}.$$

**Theorem 3.10.** Under the assumptions of Theorem 3.8, we have the inequality

$$\left|{}_aI_b^{\alpha}(fg)(x)-f(b)\ {}_aI_b^{\alpha}g(x)-g(b)\ {}_aI_b^{\alpha}f(x)-\frac{f(b)g(b)(b-a)^{\alpha}}{\Gamma(1+\alpha)}\right|$$

$$\leq \frac{\left\|f^{(\alpha)}\right\|_{\infty}\left\|g^{(\alpha)}\right\|_{\infty}}{\Gamma^2(1+\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(b-a)^{3\alpha}.$$

*Proof.* From (3.13) and (3.14), we observe that

$$f(x)g(x)-f(b)g(x)-g(b)f(x)+f(b)g(b)$$

$$= \left[\frac{1}{\Gamma(1+\alpha)}\int_x^b f^{(\alpha)}(t)\,(dt)^{\alpha}\right]\left[\frac{1}{\Gamma(1+\alpha)}\int_x^b g^{(\alpha)}(t)\,(dt)^{\alpha}\right].$$

Integrating the both sides of equality (3.15) with respect to $x$ over $[a,b]$ and using the properties

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of modulus, we get

$$\begin{aligned}
&\left| {}_aI_b^{\alpha}\left(fg\right)(x) - f(b)\ {}_aI_b^{\alpha}g(x) - g(b)\ {}_aI_b^{\alpha}f(x) - \frac{f(b)g(b)\left(b-a\right)^{\alpha}}{\Gamma\left(1+\alpha\right)} \right| \\
\leq\ & \frac{1}{\Gamma\left(1+\alpha\right)} \int_a^b \left[ \frac{1}{\Gamma\left(1+\alpha\right)} \int_x^b \left| f^{(\alpha)}(t) \right| (dt)^{\alpha} \right] \left[ \frac{1}{\Gamma\left(1+\alpha\right)} \int_x^b \left| g^{(\alpha)}(t) \right| (dt)^{\alpha} \right] (dx)^{\alpha} \\
\leq\ & \frac{\left\| f^{(\alpha)} \right\|_{\infty} \left\| g^{(\alpha)} \right\|_{\infty}}{\Gamma^3\left(1+\alpha\right)} \int_a^b \left(b-x\right)^{2\alpha} (dx)^{\alpha} \\
=\ & \frac{\left\| f^{(\alpha)} \right\|_{\infty} \left\| g^{(\alpha)} \right\|_{\infty}}{\Gamma^2\left(1+\alpha\right)} \frac{\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)} \left(b-a\right)^{3\alpha}.
\end{aligned}$$

The proof is completed. Q.E.D.

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